Component Synthesis Method for System Transient Responses

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A technique for calculating the transient responses of a system is presented. The system may be formed by joining components through the connecting elements at the boundaries of the components. The damping characteristics of the system are better represented because of the different damping factors that can be used in different components. The connecting elements at the joints between substructures can be rigid, linear, and nonlinear elements. The transient responses of the system are calculated based on the geometric compatibility and force equilibrium equations. For the rigid and linear connecting elements at the joints, no iteration is required, and direct marching procedures are used to calculate the responses. For nonlinear connecting elements, the interconnecting force vectors at the joints are determined using iterative procedures, and the responses are computed subsequently. This method is more efficient than the methods that iterate on the entire system coordinates because iterations are performed only on the boundary coordinate vectors, which are usually much smaller vectors than the entire system coordinate vectors. Numerical results indicate that the proposed method is most suitable for systems having linear subsystems with rigid, linear, and nonlinear connectivities between system components.

Nomenclature

C, K, M	= damping, stiffness, and mass matrices, respectively
$X, \dot{X}$	= displacement and velocity vectors
X_{IJ}	= response vector at the junction between substructures I and J
f	= force vector
$f(n\delta T)$	= average force vector at time $t = n\delta T$ and $(n-1)\delta T$
f_{IJ}	= interconnecting force vector at the junction between substructures I and J
f_C^I	$= f_C^I = [(f_{IJ}^I)^T, (f_{IL}^I)^T, \ldots, (f_{IK}^I)^T]^T$
$\dot{f}_C^I$	= time derivative of f_C^I
$q(t), \dot{q}(t)$	= modal coordinate vector and modal velocity vector, respectively, in time domain
$q_i(n), \dot{q}_i(n)$	= modal coordinate and modal velocity, respectively, of the ith mode at $t = n\delta T$
$q_{iC}(n), \dot{q}_{iC}(n)$	= particular solution of the ith modal coordinate and modal velocity at $t = n\delta T$ due to the interconnecting force vector f_C between substructures
$q_{iE}(n), \dot{q}_{iE}(n)$	= particular solution of the ith modal coordinate and modal velocity at $t = n\delta T$ due to the external force vector f_E
$q_{iH}(n), \dot{q}_{iH}(n)$	= homogeneous solution of the ith modal coordinate and modal velocity at $t = n\delta T$
w_i	= ith natural frequency of the system
Φ	= modal matrix $\Phi = [\phi_1, \phi_2 \ldots, \phi_N]$
Φ^T	= transpose of modal matrix
δT	= time-step increment
ϵ_i	= damping factor of the ith mode
ϕ_i	= ith eigenvector or modal vector

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Subscripts and Superscripts

I	= superscript to indicate substructure I
IJ	= subscript indicates that the matrix or vector is at the junction between substructure I and substructure J

I. Introduction

THE method commonly used for obtaining the transient responses of a linear dynamic system is the modal superposition method as discussed by Meirovitch.[1] In this method, the eigensolution of the system must be known before the analysis can proceed. The eigensolution of the system can be obtained by finite-element analysis. Computer programs such as NASTRAN and ANSYS are widely used for this purpose. One of the other methods for obtaining the eigensolution is by the component modal synthesis. The component modal synthesis technique has been studied by Hurty,[2] Hale and Meirovitch,[3] Craig and Chang,[4] and Hou.[5] To avoid the necessity of including the complete modal set for each substructure, transformation and approximation techniques were used by Rubin,[6] Hintz,[7] Guyan,[8] Geering,[9] and Kuang and Tsuei.[10] These methods also reduce the number of computations required for deriving the system eigensolution. For the nonlinear system, the modal superposition method cannot be used. The techniques discussed by Brock[11] and Butenin[12] can be utilized to obtain the nonlinear analysis of the system. Duffing[13] and Timoshenko and Young[14] studied the methods for analyzing the systems with nonlinear springs. Thomson[15] presented various approaches that can be used for solving nonlinear dynamic systems. Most of the methods required that the entire dynamic matrix of the system be used during the iteration or approximation processes.

The proposed method, which is an extension of the substructure modal synthesis method presented in Yee and Tsuei,[16] is able to overcome the deficiencies of the modal superposition method. The nonlinear analysis performed by this technique, which is also discussed in Tsuei and Yee,[17] requires fewer computations than the discussed iteration or approximation methods.

II. Derivation of the Substructure Transient Response Equations

The equation of motion of a substructure can be represented as follows:

$$M\ddot{X} + C\dot{X} + KX = f \tag{1}$$

With the modal matrix Φ, the displacement and velocity vectors X, $\dot{X}$ can be transformed to

$$X = \Phi q, \qquad \dot{X} = \Phi\dot{q} \tag{2}$$

Substituting Eq. (2) into Eq. (1), Eq. (1) becomes

$$M\Phi\ddot{q} + C\Phi\dot{q} + K\Phi q = f \tag{3}$$

Premultiply Eq. (3) by Φ^T:

$$\Phi^T M\Phi\ddot{q} + \Phi^T C\Phi\dot{q} + \Phi^T K\Phi q = \Phi^T f \tag{4}$$

With the assumption that the substructure C matrix is proportional to substructure K and M matrices, Eq. (4) can be decoupled as

$$\ddot{q}_i + 2\epsilon_i w_i \dot{q}_i + w_i^2 q_i = \phi_i^T f, \qquad i = 1,2,\ldots,N \tag{5}$$

where N is the number of degrees of freedom in the substructure.

The homogeneous solution for Eq. (5) for $t = n\delta T$ is represented by Eqs. (6) and (7).

$$q_{iH}(n) = q_i(n-1)a_i + \dot{q}_i(n-1)b_i \tag{6}$$

where

$$a_i = \exp(-\epsilon_i w_i \delta T)\left[\cos(w_{di}\delta T) + \epsilon_i(1-\epsilon_i^2)^{-1/2}\sin(w_{di}\delta T)\right]$$

$$b_i = \exp(-\epsilon_i w_i \delta T) w_{di}^{-1}\sin(w_{di}\delta T)$$

with w_{di} the damped natural frequency, and

$$\dot{q}_{iH}(n) = q_i(n-1)c_i + \dot{q}_i(n-1)d_i \tag{7}$$

where

$$c_i = \exp(-\epsilon_i w_i \delta T)\Big(\Big\{-\epsilon_i w_i[\cos(w_{di}\delta T) + \epsilon_i(1-\epsilon_i^2)^{-1/2}\sin(w_{di}\delta T)]\Big\} + \Big\{-w_{di}[\sin(w_{di}\delta T) - \epsilon_i(1-\epsilon_i^2)^{-1/2}\cos(w_{di}\delta T)]\Big\}\Big)$$

$$d_i = \exp(-\epsilon_i w_i \delta T)\Big\{[-\epsilon_i(1-\epsilon_i^2)^{-1/2}\sin(w_{di}\delta T)] + \cos(w_{di}\delta T)\Big\}$$

$q_{iH}(n)$, $\dot{q}_{iH}(n)$ are the homogeneous solutions in modal coordinates at time $t = n\delta T$.

The responses of the system due to the external excitation force vector f_E exerted between time step $(n-1)\delta T < t < n\delta T$ are defined as

$$q_{iE}(n) = \int_{(n-1)\delta T}^{n\delta T} \phi_i^T f_E(\Gamma) w_{di}^{-1}\exp[-\epsilon_i w_i(n\delta T - \Gamma)] \sin[w_{di}(n\delta T - \Gamma)]\mathrm{d}\Gamma \tag{8}$$

For small δT, Eq. (8) can be further reduced as follows:

$$q_{iE}(n) = g_i^T f_E(n) \tag{9}$$

where

$$g_i^T = \phi_i^T w_i^{-2}\Big\{[1 - \exp(-\epsilon_i w_i \delta T)\cos(w_{di}\delta T)] - [\epsilon_i(1-\epsilon_i^2)^{-1/2}\exp(-\epsilon_i w_i \delta T)\sin(w_{di}\delta T)]\Big\}$$

The $f_E(n)$ is the average of external applied force vector $f(\Gamma)$ at time $\Gamma = n\delta T$ and $\Gamma = (n-1)\delta T$, and

$$\dot{q}_{iE}(n) = \dot{g}_i^T f_E(n) + g_i^T \dot{f}_E(n) \tag{10}$$

where

$$\dot{g}_i^T = \phi_i^T w_{di}^{-1}\exp(-\epsilon_i w_i \delta T)\sin(w_{di}\delta T)$$

The effect of the time derivative of applied force vector, $\dot{f}_E(n)$, is also accounted for during the modal velocity calculations.

Similarly, the responses due to the interconnecting force vector f_C between substructures are described by Eq. (11):

$$q_{iC}(n) = g_i^T f_C(n) \tag{11}$$

The modal velocity $\dot{q}_{iC}(n)$ can be calculated by the following equation:

$$\dot{q}_{iC}(n) = \dot{g}_i^T f_C(n) + g_i^T \dot{f}_C(n) \tag{12}$$

The total responses of the substructure at $t = n\delta T$ can be written as

$$q(n) = q_H(n) + q_E(n) + q_C(n)$$
$$X(n) = \Phi q(n) \tag{13}$$

$$\dot{q}(n) = \dot{q}_H(n) + \dot{q}_E(n) + \dot{q}_C(n)$$
$$\dot{X}(n) = \Phi\dot{q}(n) \tag{14}$$

where $f_C(n)$ is the interconnecting force vectors between substructures and $\dot{f}_C(n)$ the time derivative of interconnecting force vectors between substructures.

The unknowns in Eqs. (11) and (12) are $f_C(n)$ and $\dot{f}_C(n)$, which will be determined when the substructures are jointed with other components in the system.

III. Partition of Response and Force Vectors

The response and force vectors can be partitioned as

$$X^I = \begin{bmatrix} X_{II} \\ X_{IJ} \\ \cdots \\ \cdots \\ \cdots \\ X_{IK} \end{bmatrix} \qquad f^I = \begin{bmatrix} f_{II} \\ f_{IJ} \\ \cdots \\ \cdots \\ \cdots \\ f_{IK} \end{bmatrix} \tag{15}$$

The graphical interpretation of the connecting terminology is shown in Fig. 1.

The purpose of such a partition, which will be demonstrated in later sections, is to facilitate the components coupling process and to reduce the system coordinates to the joint coordinates between substructures.

IV. Assembly of the System from Components

Two substructures are jointed together according to their geometric compatibility and force equilibrium equations to form a system. Response vectors of substructures I and J are partitioned into subvectors according to the procedures discussed in Section III.

The compatibility and equilibrium equations for the displacement vector are

$$K_{IJ}\left[X_{JI}^J - X_{IJ}^I\right] = f_{IJ}^I$$
$$f_{IJ}^I = -f_{JI}^J$$
$$f_C^I = f_{IJ}^I \tag{16}$$

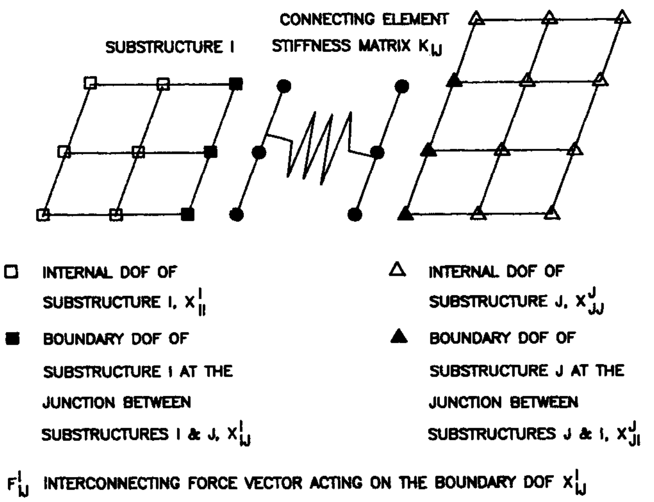

Fig. 1 Illustration of connecting terminology.

The force vector f^I_C is reduced to f^I_{IJ} because the substructure J is the only substructure attached to the substructure I.

Equations (13) and (16) are simplified, and the following equation is derived.

$$f^I_{IJ} = H\left\{\Phi^J_{JI}[q^J_H(n) + q^J_E(n)] - \Phi^I_{IJ}[q^I_H(n) + q^I_E(n)]\right\} \tag{17}$$

where

$$H = \left\{\Phi^I_{IJ}[G^I_{IJ}]^T + \Phi^J_{JI}[G^J_{JI}]^T + K_{IJ}{}^{-1}\right\}^{-1}$$

$$G^I_{IJ} = [g_1, g_2, \ldots, g_n]^I_{IJ}$$

g_i is defined in Eq. (9), and K_{IJ} is the stiffness matrix connecting substructures I and J, as shown in Fig. 1.

With the f^I_{IJ} determined, modal coordinate vector $q(n)$ and the response vector $X(n)$ for substructures can be obtained from Eq. (13). It is noted that only substructure boundary degrees of freedom are used in the calculations; thus, the final number of simultaneous equations are much reduced.

Another set of compatibility and equilibrium equations are used to calculate the velocity of the system.

$$K_{IJ}[\dot{X}^J_{JI} - \dot{X}^I_{IJ}] = \dot{f}^I_{IJ}$$

and

$$\dot{f}^I_{IJ} = -\dot{f}^J_{JI}, \qquad \dot{f}^I_C = \dot{f}^I_{IJ} \tag{18}$$

Equation (14) is further simplified as

$$\dot{q}^I = \dot{q}^I_H + \dot{q}^I_E + [G^I]^T\dot{f}^I_C + [\dot{G}^I]^T f^I_C$$

$$\dot{X}^I_{IJ} = \Phi^I_{IJ}\dot{q}^I \tag{19}$$

where $\dot{G}^I = [\dot{g}_1, \dot{g}_2, \ldots, \dot{g}_n]^I$ and $\dot{g}_i$ is defined in Eq. (10).

From Eqs. (18) and (19), the unknown $\dot{f}^I_{IJ}$ is expressed as

$$\dot{f}^I_{IJ} = H\left\{\Phi^J_{JI}\left[\dot{q}^J_H + \dot{q}^J_E + (\dot{G}^J)^T f^J_C\right]\right.$$
$$\left. - \Phi^I_{IJ}\left[\dot{q}^I_H + \dot{q}^I_E + (\dot{G}^I)^T f^I_C\right]\right\} \tag{20}$$

The modal velocity vector $\dot{q}^I$ and the physical velocity vector $\dot{X}^I$ can be obtained from Eq. (19).

The procedure just discussed gives the response of each substructure at time $t = n\delta T$ based on the initial conditions and prescribed force functions at time $t = (n-1)\delta T$. The same procedure can be repeated to obtain the system responses at time $t = (n+1)\delta T$.

V. Discussion of the Solution Schemes for the Rigid, Linear, and Nonlinear Connectivities Between System Components

The connecting element between substructures can be classified into three types, which are now discussed.

A. Rigid Connecting Element

For the rigid connections between substructures, the K_{IJ} term in Eqs. (16-20) are eliminated, such that

$$X^I_{IJ} = X^J_{JI}$$

$$H = \left[\Phi^I_{IJ}(G^I_{IJ})^T + \Phi^J_{JI}[G^J_{JI}]^T\right]^{-1}$$

$$\dot{X}^I_{IJ} = \dot{X}^J_{JI}$$

Other than the preceding modifications, the procedures are identical to those discussed in Section IV.

B. Linear Connecting Element

The equations and procedures discussed in Section IV are for linear connections. Therefore, no modification is needed.

C. Nonlinear Connecting Element

The interconnecting stiffness matrix K_{IJ} between subsystems is nonlinear and is changed with time. The following iterative procedures are used to calculate the system transient responses. For responses at $t = n\delta T$:

Step 1: Use the K_{IJ} at the previous time step $t = (n-1)\delta T$.

Step 2: Calculate the joint force vector f^I_{IJ} from Eq. (17).

Step 3: Obtain the displacement vectors X^I_{IJ} and X^J_{JI} from Eq. (13) and calculate the relative displacement vector $|X^J_{JI} - X^I_{IJ}|$.

Step 4: Use the calculated relative displacement vector to obtain matrix $\hat{K}_{IJ}$ from the connecting element stiffness specification.

Step 5: Compare the value K_{IJ} and $\hat{K}_{IJ}$. If the Euclidean norm, $||\hat{K}_{IJ} - K_{IJ}||^2$, is less than the tolerance limit, go to step 6. Otherwise, use the following equations:

$$\bar{K}_{IJ} = (\hat{K}_{IJ} + K_{IJ})/2$$

$$K_{IJ} = \bar{K}_{IJ}$$

Use the updated stiffness matrix K_{IJ}, and repeat steps 1-5. If the stiffness matrix K_{IJ} does not converge after the specified number of iterations, reduce the time step, and repeat steps 1-5.

Step 6: Calculate the $\dot{q}(n)$ and $\dot{X}(n)$ from Eqs. (18-20).

Step 7: Repeat steps 1-6 for the next time step $t = (n+1)\delta T$.

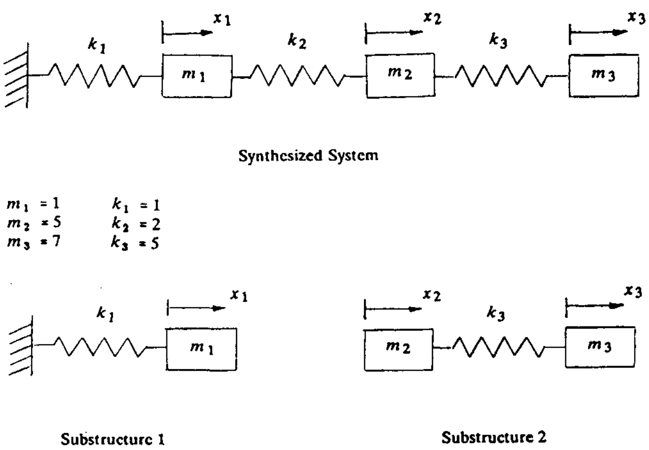

Fig. 2 A lumped mass-spring system and its substructures.

VI. Numerical Results and Examples

Example 1

A three-degree-of-freedom lumped mass-spring system as shown in Fig. 2, is analyzed. The spring K_2 is used to connect subsystems 1 and 2. The stiffness K_2 is constant and is equal to 2. A step force function with magnitude 10 is exerted on location X_1. The synthesized and the close form solutions of the system are calculated. The transient responses at the X_2 location are plotted in Fig. 3. From Fig. 3, it can be seen that the synthesis method can accurately calculate the transient responses of a system.

Example 2

A clamped-clamped beam, as shown in Fig. 4, is analyzed. Substructures 1 and 2 are clamped-free beams, and they are connected by a massless beam element. Eight substructure modes for each substructure are included for synthesis. A force with a magnitude 10 and angular frequency of 20 rad/s is applied at the Y_1 location. The system transient responses are also derived by the modal superposition method. The calculations of the modal superposition analysis are based on the eight system normal modes. The transient responses at the Y_1 position calculated by the synthesis and the modal superposition methods are compared and are plotted in Fig. 5.

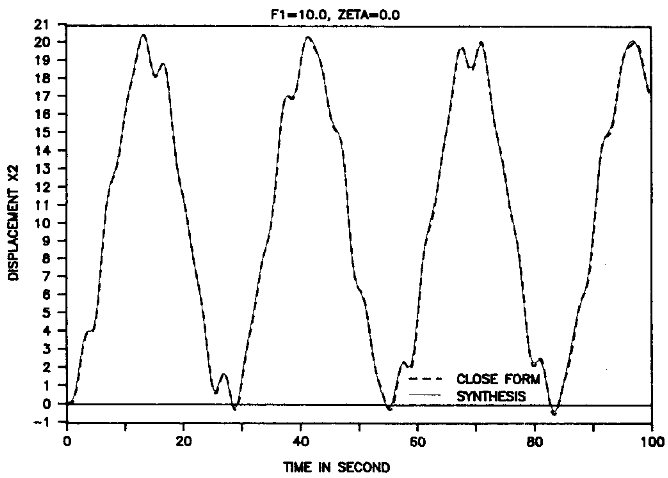

Fig. 3 Transient response of displacement X_2 due to step force function ($F_{X1} = 10$) at X_1 location.

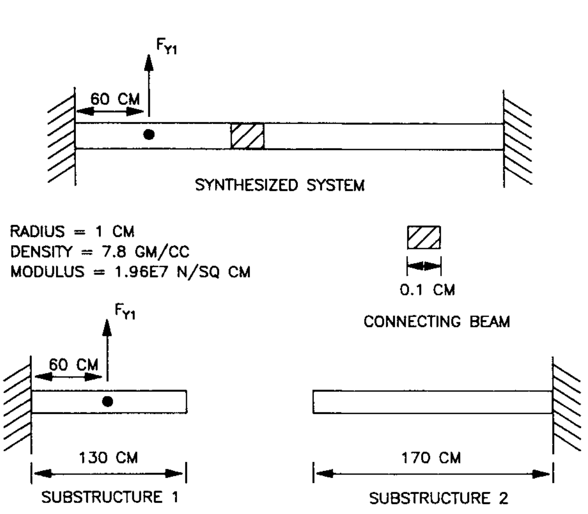

Fig. 4 A clamped-clamped beam structure and its components.

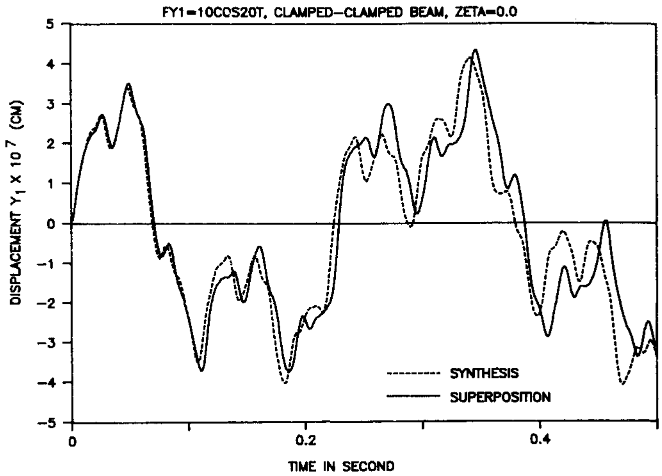

Fig. 5 Transient response at Y_1 position due to force function $F_{Y1} = 10\cos 20t$ at Y_1 location.

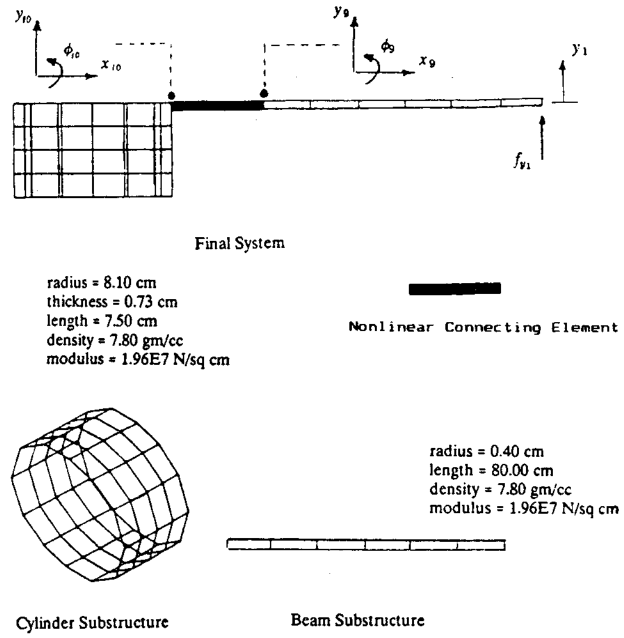

Fig. 6 The antenna structure and its components.

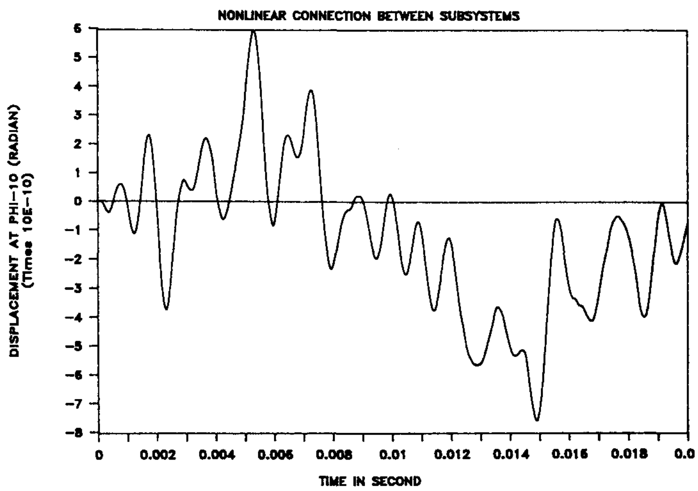

Fig. 7 Transient response at ϕ_{10} position due to force function $F_{Y1} = 10\cos 3000t$ at Y_1 location.

Example 3

An antenna structure, as shown in Fig. 6, is analyzed using the proposed method. A massless beam element with nonlinear elastic property, described in Table 1, is used to connect substructures 1 and 2. A force with a magnitude of 10 and frequency of 3000 rad/s is applied at the Y_1 location of the system. For the cylinder substructure, nine substructure modes are used, and these modes include three rigid-body modes, four prismatic modes, and two nonprismatic modes. For the beam substructure, nine substructure modes are used, and these modes include three rigid-body modes and six flexible modes. The substructure modal damping factors for the cylinder and the beam substructures are set to 0.5%. The responses at the ϕ_{10} position of the system are plotted in Fig. 7. The angular strain, which is defined as $\text{abs}(\phi_{10} - \phi_9)$ divided by the length of the connecting element (L), is plotted against the time in Fig. 8.

To further demonstrate the method, the modulus of elasticity E of the connecting element is plotted against the angular strain $\text{abs}(\phi_{10} - \phi_9)/L$, at every time step, and the curve is shown in Fig. 9. The results from the analysis indicate that, at every time step, the relationship of the modulus of elasticity E and the displacements of the substructures satisfy the given nonlinear connecting element specifications in Table 1. This validates the nonlinear analysis of the antenna structure.

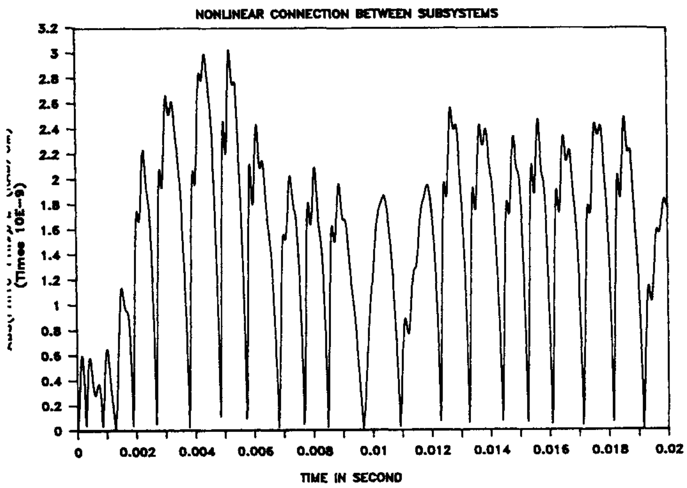

Fig. 8 Transient response of the angular strain, $(|\phi_{10} - \phi_9|)/L$, due to force function $F_{Y1} = 10\cos 3000t$ at Y_1 location.

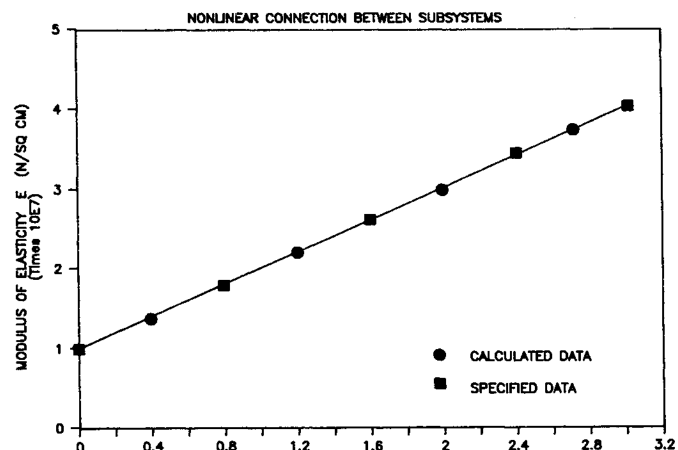

Fig. 9 Modulus of elasticity (E) vs angular strain, $(|\phi_{10} - \phi_9|)/L$, due to force function $F_{Y1} = 10\cos 3000t$ at Y_1 location.

Table 1 Specification of a nonlinear connecting element

radius	$R = 0.40$ cm
length	$L = 0.10$ cm
area	$A = 0.50265$ cm^2
inertia	$I = 0.020106$ cm^4
modulus	$E = 1.0 \times 10^7 + (1.0 \times 10^{16})(\lvert \phi_{10} - \phi_9 \rvert)/L$ N/cm^2
K matrix	

$$K = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

VII. Conclusion

A method for calculating the transient responses of a system is presented. The method is an extension of the component modal synthesis method and can be used for the linear and nonlinear dynamic systems. Its advantages are summarized as follows:

1) The eigensolutions and damping factors of the synthesized system are not required.

2) The calculation is based on the substructure modal parameters. For a given set of substructure modes, this provides an efficient method to synthesize the transient response of a system.

3) The damping factors for each component of the system can be defined independently. Thus, the damping of the system can be better represented than in the modal superposition method.

4) The method condenses the system coordinates to the physical degrees of freedom at the joint locations between substructures, and the other system coordinates can be recovered by simple multiplication.

5) For the system without any nonlinearity, the matrix H in Eqs. (17-20), a_i and b_i in Eq. (6), and c_i and d_i in Eq. (7) are required to be calculated only once. Responses at each time step are calculated by simple marching procedures.

6) The eigenvalues and eigenvectors of the substructures can be stored in the computer, and a new system configuration can be readily formed.

7) Nonlinear connecting elements between system components can be included for analysis. In such cases, iteration is performed on the coordinate vectors at the boundaries between substructures. This is much more efficient than the methods that iterate on the entire system coordinates.

8) The method is readily applied for the parametric study of the transient responses of a system.

Several structures are analyzed, and three examples are chosen for discussion. Numerical results indicate that the suggested method is most suitable for the systems that can be divided into subsystems with linear and nonlinear connecting elements between substructures.

It should be noted that the iteration procedures described previously can be further improved. For a system with many nonlinear connecting elements, a more sophisticated matrix convergence procedure should be used. Some of these procedures are discussed in Young[18] and Varga.[19]

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